DEPTH ZERO REPRESENTATIONS OVER $\overline{\mathbb{Z}}[\frac{1}{p}]$

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Abstract. We consider the category of depth 0 representations of a p-adic quasi-split reductive group with coefficients in $\overline{\mathbb{Z}}[\frac{1}{p}]$. We prove that the blocks of this category are in natural bijection with the connected components of the space of tamely ramified Langlands parameters for G over $\overline{\mathbb{Z}}[\frac{1}{p}]$. As a particular case, this depth 0 category is thus indecomposable when the group is tamely ramified. Along the way we prove a similar result for finite reductive groups. We then outline a potential application to the Fargues-Scholze and Genestier-Lafforgue semisimple local Langlands correspondences. Namely, contingent on a certain "independence of ℓ" property, our results imply that these correspondences take depth 0 representations to tamely ramified parameters.

1. Main results

We prove two results on the representation theory of finite reductive groups and on that of p-adic reductive groups. We first state these results and then explain our motivations and some connections to the existing literature.

1.0.1. **Theorem** (Theorem 2.0.1). *Let $\mathbf{G}$ be a reductive group over $\overline{\mathbb{F}}_p$ and F the Frobenius map associated to a $\mathbb{F}_{p^r}$-rational structure for some $r \geqslant 1$. Then the category $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(\mathbf{G}^{\mathsf{F}})$ is indecomposable. Equivalently, the central idempotent 1 in $\overline{\mathbb{Z}}[\frac{1}{p}]\,\mathbf{G}^{\mathsf{F}}$ is primitive.*

This result initially appeared as one step in our study of the p-adic case below. We have decided to single it out because the statement is simple and quite natural. It might be an interesting problem to try and devise criteria for a similar statement to hold true for an abstract finite group G and a prime divisor p of $|G|$.

Let now $\mathbf{G}$ be a reductive group over a local non-archimedean field F with residue field $k_F := \mathbb{F}_{p^r}$. We put $G := \mathbf{G}(F)$. For any commutative ring R in which p is invertible, we denote by $\mathrm{Rep}_R(G)$ the category of smooth RG-modules. The Bernstein center $\mathfrak{Z}_R(G)$ is by definition the center of this category. We refer to subsection 3.1 for the definition of *depth* 0 smooth RG-modules. They form a direct factor subcategory $\mathrm{Rep}^0_R(G)$, which correspond to some idempotent $\varepsilon_0 \in \mathfrak{Z}_R(G)$. The following statement is a sample of what we prove about $\mathrm{Rep}^0_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$.

1.0.2. **Theorem** (Theorem 3.6.1). *Suppose that $\mathbf{G}$ is quasi-split and tamely ramified over F. Then the abelian category $\mathrm{Rep}^0_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ is indecomposable. Equivalently, ε_0 is a primitive idempotent of $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$.*

This result was mainly inspired by its "dual" counterpart in [DHKM20], where the moduli space $Z^1(W_F^0, \hat{\mathbf{G}})$ of Langlands parameters for $\mathbf{G}$ was constructed over $\overline{\mathbb{Z}}[\frac{1}{p}]$ and studied. Concretely, $Z^1(W_F^0, \hat{\mathbf{G}})$ classifies 1-cocycles $W_F^0 \longrightarrow \hat{\mathbf{G}}$ where :

- $\hat{\mathbf{G}}$ denotes the dual group of $\mathbf{G}$, considered as a split pinned reductive group scheme over $\overline{\mathbb{Z}}[\frac{1}{p}]$, and endowed with an action of the Galois group Γ_F of F that preserves the pinning

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- $W_F^0 \subset W_F \subset \Gamma_F$ is some modification of the Weil group of F.

In [DHKM20], this moduli space is decomposed according to restriction of 1-cocycles to the wild inertia subgroup $P_F \subset W_F^0$. In particular, the "tame" summand $Z^1(W_F^0, \hat{\mathbf{G}})_{\rm tame}$ parametrizes 1-cocycles whose restriction to P_F is locally (for the étale topology) conjugate to the trivial cocycle. According to [DHKM20, Thm 4.29], this summand is connected, provided that $\mathbf{G}$ is tamely ramified. Since tame parameters are supposed to correspond to depth 0 representations by any form of local Langlands correspondence, we like to see the last theorem as the group side analogue of this connectedness result on the parameter side. Interestingly, the proof of [DHKM20, Thm 4.29] consists in, first, classifying the connected components of $Z^1(W_F^0, \hat{\mathbf{G}})_{\overline{\mathbb{Z}}_\ell, {\rm tame}}$ for each $\ell \neq p$, and then, using different ℓ's to get the result. Similarly, one way to formulate the indecomposability of $\mathrm{Rep}^0_{\mathbb{Z}[\frac{1}{p}]}(G)$ is as follows (see Remark 3.3.4).

1.0.3. **Corollary.** *Under the same hypothesis on $\mathbf{G}$, given π, π' two irreducible $\overline{\mathbb{Q}}G$-modules of depth 0, there is a sequence of primes $\ell_1, \cdots, \ell_r$ and a sequence $\pi_0 = \pi, \pi_1, \cdots, \pi_r = \pi'$ of irreducible $\overline{\mathbb{Q}}G$-modules such that π_{i-1} and π_i belong to the same ℓ_i-block for each $i = 1, \cdots, r$.*

We note that Sécherre and Stevens have used in [SS19] a similar statement in the context of inner forms of $GL(n)$ (but for arbitrary "endoclasses") in order to gain control on the Jacquet-Langlands correspondence for *complex* representations. This provides a striking example how to use this kind of results for problems a priori unrelated to congruences. In 1.1 below, we outline another kind of application.

Meanwhile, let us observe that it is not always true, even for $\mathbf{G}$ a torus, that the tame summand of $Z^1(W_F^0, \hat{\mathbf{G}})$ is connected. In Theorem 3.7.3, we work out the decomposition of $Z^1(W_F^0, \hat{\mathbf{G}})_{\rm tame}$ into connected components for general $\mathbf{G}$, and we prove in particular that these connected components are simply transitively permuted by a certain abelian p-group of "central cocycles". On the other hand, in Theorem 3.6.2, we work out the decomposition of $\mathrm{Rep}^0_{\mathbb{Z}[\frac{1}{p}]}(G)$ into a product of blocks for quasi-split G, and prove that these blocks are simply transitively permuted by a certain abelian p-group of characters of G. After identifying these two p-groups and matching the principal component with the principal block, we then conclude:

1.0.4. **Theorem.** *Assume $\mathbf{G}$ is quasi-split over F. Then there is a natural bijection between connected components of $Z^1(W_F^0, \hat{\mathbf{G}})_{\rm tame}$ and blocks of $\mathrm{Rep}^0_{\mathbb{Z}[\frac{1}{p}]}(G)$.*

Again, this implies that, under the same quasi-splitness hypothesis, a $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$ of depth 0 can be connected to a depth 0 character of p-power order through a sequence of "congruences" modulo different primes.

For a non quasi-split group $\mathbf{G}$, there is in general an additional "relevance" condition on 1-cocycles for them to provide Langlands parameters of G. Although this relevance condition might mess up with connected components, we believe it does not actually happen, i.e. the above result should be true with no quasi-split assumption. As an evidence for this expectation, we prove :

1.0.5. **Theorem** (Corollary 3.4.4). *Suppose that p does not divide $|\pi_1(\mathbf{G}_{\rm der})|$ and that the torus $\mathbf{G}_{\rm ab}$ is P_F-induced. Then $\mathrm{Rep}^0_{\mathbb{Z}[\frac{1}{p}]}(G)$ is indecomposable.*

This is in accordance with the fact that, if p does not divide $|\pi_0(Z(\hat{\mathbf{G}}))|$ and $(Z(\hat{\mathbf{G}})^\circ)^{P_F}$ is connected, then $Z^1(W_F^0, \hat{\mathbf{G}})_{\rm tame}$ is connected.

1.1. **Sample (potential) application.** Among the major recent breakthrough towards constructing the conjectural local Langlands correspondence for a p-adic group G, Fargues and Scholze [FS21] have recently used new geometric tools to attach to any irreducible representation π of G, a semisimple local Langlands parameter φ_π. Their construction is compatible with parabolic induction, so that, for example, the semisimple parameter attached to a principal series is unramified as expected. However it is in general very difficult to say anything on this parameter, especially when π is cuspidal. For example, as alluded to above, a natural expectation is that if π has depth 0, then φ_π is tamely ramified.

Fargues and Scholze's construction actually provides a map

$$\mathrm{FS}_\ell :\ \mathcal{O}(Z^1(W_F^0,\hat{\mathbf{G}})_{\overline{\mathbb{Z}}_\ell})^{\hat{\mathbf{G}}} \longrightarrow \mathfrak{Z}_{\overline{\mathbb{Z}}_\ell}(G)$$

for each prime $\ell \neq p$. The $\hat{\mathbf{G}}(\overline{\mathbb{Q}}_\ell)$ conjugacy class of semisimple parameters φ_π associated to $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)$ is then given by the $\overline{\mathbb{Q}}_\ell$-point of $Z^1(W_F^0,\hat{\mathbf{G}}) /\!\!/ \hat{\mathbf{G}}$ obtained by composing FS_ℓ with the infinitesimal character $\mathfrak{Z}_{\overline{\mathbb{Z}}_\ell}(G) \longrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\pi) = \overline{\mathbb{Q}}_\ell$. In particular, this construction is compatible with congruences mod ℓ. Thanks to the description of the connected components of $Z^1(W_F^0,\hat{\mathbf{G}}))_{\overline{\mathbb{Z}}_\ell}$ in [DHKM20, Thm 4.8], this implies for example that if π,π' belong to the same block of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$, then the restrictions of φ_π and $\varphi_{\pi'}$ to the prime-to-ℓ inertia subgroup I_F^ℓ coincide.

In order to control φ_π for a depth 0 irreducible representation π, one would like to use congruences modulo different primes, as encapsulated in our main results above. However, one important open problem in Fargues and Scholze's construction is to prove "independence of ℓ". For example, if $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$, we want φ_π to be a semisimple $\overline{\mathbb{Q}}$-parameter independent on the choice of an ℓ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$. In other words, we expect that the family of maps FS_ℓ for $\ell \neq p$ is induced by a map $\mathcal{O}(Z^1(W_F^0,\hat{\mathbf{G}}))^{\hat{\mathbf{G}}} \xrightarrow{\mathrm{FS}} \mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. If such a map FS exists, then our results imply that, at least for $\mathbf{G}$ quasi-split, φ_π is tamely ramified whenever π has depth 0.

A similar discussion applies to the construcion of Genestier and Lafforgue [GL18]. Actually, there is a bit more hope to prove independence of ℓ in their setting. Indeed this question reduces to supercuspidal π's and these representations globalize to automorphic representations to which Lafforgue's global theory applies. The latter global theory uses ℓ-adic cohomology of algebraic varieties (stacks actually) for which some independence of ℓ results are known, in contrast to diamonds. However not enough to get what we need here, to our knowledge.

2. Finite groups

Let $\mathbf{G}$ be a reductive group over $\overline{\mathbb{F}}_p$ and F the Frobenius map associated to a $\mathbb{F}_q$-rational structure on $\mathbf{G}$, where $q = p^r$ for some $r \geqslant 1$. The goal of this section is to prove the following theorem.

2.0.1. **Theorem.** *The category* $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(\mathbf{G}^{\mathsf{F}})$ *is indecomposable. Equivalently, the central idempotent* 1 *in* $\overline{\mathbb{Z}}[\frac{1}{p}]\,\mathbf{G}^{\mathsf{F}}$ *is primitive.*

2.1. **General facts on blocks of finite groups.** Let us start with an abstract finite group G. For any commutative ring Λ, denote by $\mathrm{Rep}_\Lambda G$ the abelian category of ΛG-modules. Recall that a block of ΛG is an indecomposable summand of the category $\mathrm{Rep}_\Lambda G$. These blocks correspond bijectively to indecomposable two-sided ideals of the ring ΛG and to primitive ideals in the center $Z(\Lambda G)$ of the ring ΛG, which is also the center of the category $\mathrm{Rep}_\Lambda G$.

For a prime ℓ, Hensel's lemma implies that the reduction map $\overline{\mathbb{Z}}_\ell G \longrightarrow \overline{\mathbb{F}}_\ell G$ induces a bijection on primitive idempotents, whence a bijection between blocks of

$\overline{\mathbb{F}}_\ell G$ and of $\overline{\mathbb{Z}}_\ell G$. The decomposition of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell} G$ as a direct sum of blocks induces a partition of the set $\mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(G)$ of isomorphism classes of simple $\overline{\mathbb{Q}_\ell} G$-modules. Since any automorphism of $\overline{\mathbb{Q}_\ell}$ is ℓ-adically continuous, this partition is stable under such an automorphism. It follows that it can be transported unambiguously to a partition of the set $\mathrm{Irr}(G)$ of irreducible complex representations of G. Each factor set occurring in this partition will be called an ℓ-block of $\mathrm{Irr}(G)$.

Here is another point of view on ℓ-blocks of $\mathrm{Irr}(G)$. In $\mathbb{C}G$ we have the decomposition $1 = \sum_{\pi\in\mathrm{Irr}(G)} e_\pi$ of 1, where $e_\pi = \frac{\dim\pi}{|G|}\sum_{g\in G} \mathrm{tr}(\pi(g))g$ are the primitive central idempotents of $\mathbb{C}G$. As the formula shows, each e_π belongs to $\overline{\mathbb{Z}}[\frac{1}{|G|}]G$ so that this decomposition of 1 actually holds in $\overline{\mathbb{Z}}[\frac{1}{|G|}]G$. Denote by $|G|^{\ell'}$ the prime-to-ℓ factor of $|G|$, and declare that a subset $I \subset \mathrm{Irr}(G)$ is ℓ-integral if $\sum_{\pi\in I} e_\pi \in \overline{\mathbb{Z}}[\frac{1}{|G|^{\ell'}}]G$. Clearly, ℓ-integral subsets of $\mathrm{Irr}(G)$ are stable under taking unions, intersections and complementary subsets.

2.1.1. **Lemma.** *The ℓ-blocks of $\mathrm{Irr}(G)$ are the minimal non-empty ℓ-integral subsets.*

Proof. Since both the property of being an ℓ-block and of being ℓ-integral are invariant under field automorphisms, we may transport the statement to $\mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(G)$ where it follows from the fact that for any block B of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell} G$, the corresponding primitive central idempotent e_B in $Z(\overline{\mathbb{Z}}_\ell G)$ is given by $e_B = \sum_{\pi\in\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G)\cap B} e_\pi$. $\square$

Let us denote by $\sim_\ell$ the equivalence relation on $\mathrm{Irr}(G)$ whose equivalence classes are the ℓ-blocks. Now, fix a prime p, and denote by $\sim$ the equivalence relation generated by all $\sim_\ell$ for $\ell \neq p$. Explicitly, we thus have $\pi \sim \pi'$ if and only if there exist ℓ_1,...,ℓ_r a sequence of primes different from p and $\pi_1, \cdots, \pi_{r-1} \in \mathrm{Irr}(G)$ such that

$$\pi \sim_{\ell_1} \pi_1 \sim_{\ell_2} \pi_2 \sim_{\ell_3} \cdots \sim_{\ell_r} \pi'.$$

2.1.2. **Proposition.** *The $\sim$-equivalence classes are the minimal non-empty subsets $I \subset \mathrm{Irr}(G)$ such that $e_I := \sum_{\pi\in I} e_\pi \in \overline{\mathbb{Z}}[\frac{1}{p}]G$. Moreover, the map $I \mapsto e_I$ is a bijection from $\mathrm{Irr}(G)/\sim$ onto the set of primitive idempotents of $\overline{\mathbb{Z}}[\frac{1}{p}]G$.*

Proof. Follows from the previous lemma and the equality $\overline{\mathbb{Z}}[\frac{1}{p}] \cap \overline{\mathbb{Z}}[\frac{1}{|G|}] = \bigcap_{\ell\neq p} \overline{\mathbb{Z}}[\frac{1}{|G|^{\ell'}}]$. $\square$

Specializing our discussion to the case $G = \mathbf{G}^{\mathsf{F}}$, the last proposition shows that proving Theorem 2.0.1 is equivalent to proving that there is only one $\sim$-equivalence class in $\mathrm{Irr}(\mathbf{G}^{\mathsf{F}})$. Before doing so, we need a recollection of Deligne-Lusztig theory.

2.2. **Blocks of finite reductive groups.** Fix a reductive group $(\mathbf{G}^*, \mathsf{F}^*)$ over $\mathbb{F}_q$ that is dual to $(\mathbf{G}, \mathsf{F})$ in the sense of Lusztig. Using their "twisted" induction functors, Deligne and Lusztig define a partition $\mathrm{Irr}(\mathbf{G}^{\mathsf{F}}) = \bigsqcup_s \mathcal{E}(\mathbf{G}^{\mathsf{F}}, s)$ of irreducible representations into "Deligne-Lusztig series" associated to semisimple elements s of $\mathbf{G}^{*\mathsf{F}^*}$ up to $\mathbf{G}^{*\mathsf{F}^*}$-conjugacy. Note that this partition depends on certain compatible choices of roots of unity, but these choices will be irrelevant to our matters.

For a prime $\ell \neq p$ and a semisimple element $s \in \mathbf{G}^{*\mathsf{F}^*}$ of order prime to ℓ, we put

$$\mathcal{E}_\ell(\mathbf{G}^{\mathsf{F}}, s) := \bigcup_{t_{\ell'}=s} \mathcal{E}(\mathbf{G}^{\mathsf{F}}, t),$$

where $t_{\ell'}$ denotes the prime-to-ℓ part of t (i.e., we write $t = t_{\ell'} t_\ell$ in $\langle t\rangle$, with t_ℓ of order ℓ and $t_{\ell'}$ of order prime to ℓ). The following fundamental results are due to Broué and Michel, resp. Hiss, and are stated in [CE04, Thm. 9.12].

(1) $\mathcal{E}_\ell(\mathbf{G}^{\mathsf{F}}, s)$ is a union of ℓ-blocks.

(2) For each ℓ-block B such that $\mathrm{Irr}(\mathbf{G}^{\mathsf{F}}, B) \subseteq \mathcal{E}_\ell(\mathbf{G}^{\mathsf{F}}, s)$, one has $\mathrm{Irr}(\mathbf{G}^{\mathsf{F}}, B) \cap \mathcal{E}(\mathbf{G}^{\mathsf{F}}, s) \neq 0$.

From these results we easily deduce the following fact.

2.2.1. **Proposition.** *Any representation π in $\mathrm{Irr}(\mathbf{G}^{\mathsf{F}})$ is $\sim$-equivalent to a representation in $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$.*

Proof. Let s be a semisimple element in ${\mathbf{G}^*}^{\mathsf{F}^*}$ such that $\pi \in \mathcal{E}(\mathbf{G}^{\mathsf{F}}, s)$, and let ℓ be a prime that divides the order of s. Note that $\ell \neq p$. As above, denote by $s_{\ell'}$ the prime-to-ℓ part of s. Then (2) above tells us that the ℓ-block containing π intersects $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, s_{\ell'})$. Therefore, the $\sim$-equivalence class of π intersects $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, s_{\ell'})$ too. But the order of $s_{\ell'}$ is the prime-to-ℓ factor of the order of s. So, arguing inductively on the number of prime divisors of the order of s, we conclude that the $\sim$-equivalence class of π intersects $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$. □

Let us now denote by $\sim^1$ the equivalence relation on $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$ defined in the same way as $\sim$, with every intermediate representation π_i taken in $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$. By the last proposition, in order to prove Theorem 2.0.1, it suffices to show that $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$ has a unique $\sim^1$-equivalence class.

2.2.2. **Proposition.** *Let $\mathbf{G}_{\mathrm{ad}}$ be the adjoint group of $\mathbf{G}$, denote by $\pi : \mathbf{G}^{\mathsf{F}} \longrightarrow \mathbf{G}_{\mathrm{ad}}^{\mathsf{F}}$ the natural map and by π^* the associated pullback on representations. Then π^* induces a bijection on unipotent representations $\mathcal{E}(\mathbf{G}_{\mathrm{ad}}^{\mathsf{F}}, 1) \xrightarrow{\sim} \mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$ that is compatible with $\sim^1$-equivalence on both sides.*

Proof. The fact that π^* induces a bijection on unipotent representations is clear from the very definition of these representations. Moreover, for a prime ℓ different from p, [CE04, Thm. 17.1] tells us that $\mathbf{G}^{\mathsf{F}}$ and ${\mathbf{G}_{\mathrm{ad}}}^{\mathsf{F}}$ have the same number of unipotent ℓ-blocks and that $\pi^* : \mathbb{Z}\mathcal{E}_\ell({\mathbf{G}_{\mathrm{ad}}}^{\mathsf{F}}, 1) \longrightarrow \mathbb{Z}\mathcal{E}_\ell(\mathbf{G}^{\mathsf{F}}, 1)$ preserves the orthogonal decomposition induced by ℓ-blocks. It follows that the bijection $\pi^* : \mathcal{E}(\mathbf{G}_{\mathrm{ad}}^{\mathsf{F}}, 1) \xrightarrow{\sim} \mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$ on unipotent representations is compatible with the respective ℓ-block partitions. Since the $\sim^1$-equivalence classes are the subsets which are stable under $\sim_\ell$-equivalence for $\ell \neq p$ and minimal for this property, π^* is also compatible with the partition into $\sim^1$-equivalence classes. □

This proposition allows us to reduce the general case to the case where $\mathbf{G}$ is of adjoint type. But a group of adjoint type is a direct product of restriction of scalars of simple groups. So, in the sequel we may restrict attention to simple groups. It turns out that in some cases, there is a quick argument using 2-block theory.

2.2.3. **Theorem.** *Suppose q is odd and $\mathbf{G}$ has type $\mathbf{A}_n$, ${}^2\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ or ${}^2\mathbf{D}_n$. Then $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$ is composed of only one $\sim^1$-equivalence class.*

Proof. Indeed, by [CE04, Thm. 21.14], $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$ is included in the principal 2-block. So all the unipotent representations are already $\sim_2$-equivalent. □

In order to deal with q even and the exceptional groups, we need to recall more results about blocks that contain a unipotent representation.

2.3. *d*-**series.** The unipotent ℓ-blocks can be obtained using d-Harish-Chandra theory, which provides a partition of $\mathcal{E}(\mathbf{G}^{\mathsf{F}}, 1)$ into *d-series*, and where $d \geqslant 1$ is an integer. When $d = 1$, the 1-series are the usual Harish-Chandra series constructed via parabolic induction. In general, they are defined through an analogous pattern, relying on Deligne-Lusztig induction and the following definitions.

- An F-stable Levi subgroup of $\mathbf{G}$ is called a *d-split Levi subgroup* if it is the centralizer of a Φ_d-torus, i.e. an F-stable torus $\mathbf{S}$ such that $|\,\mathbf{S}^{\mathsf{F}^n}| = \Phi_d(q^n)$ for a certain $a \geqslant 1$ and all $n > 0$ with $n \equiv 1 \pmod a$.

- An irreducible representation $\pi \in \mathrm{Irr}(\mathbf{G}^{\mathsf{F}})$ is called *d-cuspidal* if for all proper d-split Levi subgroups $\mathbf{L}$ and all parabolic subgroups $\mathbf{P}$ with Levi $\mathbf{L}$, the Deligne-Lusztig twisted restriction ${}^{*}R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}[\pi]$ vanishes.
- A *d-cuspidal pair* for $\mathbf{G}$ is a pair $(\mathbf{L},\sigma)$ with $\mathbf{L}$ a d-split Levi subgroup of $\mathbf{G}$ and $\sigma \in \mathrm{Irr}(\mathbf{L}^{\mathsf{F}})$ d-cuspidal.
- The *d-cuspidal support* of $\pi \in \mathrm{Irr}(\mathbf{G}^{\mathsf{F}})$ is the set of all d-cuspidal pairs $(\mathbf{L},\sigma)$ such that π appears with non-zero multiplicity in the virtual character $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}[\sigma]$.

According to [BMM93, Thm 3.2], the d-cuspidal support of any unipotent $\pi \in \mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ consists of a single $\mathbf{G}^{\mathsf{F}}$-conjugacy class of d-cuspidal pairs $(\mathbf{L},\sigma)$. We thus get a partition of $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ labelled by conjugacy classes of unipotent d-cuspidal pairs. The summands appearing in this partition are called *d-series*. The remarkable relevance of d-series to the study of ℓ-blocks is summarized in the following statement.

2.3.1. **Theorem** ([CE94, Thm. 4.4]). *Let ℓ be a prime not dividing q and let d be the order of q in $\mathbb{F}_{\ell}^{\times}$. We assume that ℓ is odd, good for $\mathbf{G}$, and $\ell \neq 3$ if ${}^{3}\mathbf{D}_4$ is involved in $(\mathbf{G},\mathsf{F})$. Then, the map $B \mapsto B \cap \mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ induces a bijection*

$$\{\ell\text{-blocks } B \subset \mathcal{E}_{\ell}(\mathbf{G}^{\mathsf{F}},1)\} \xrightarrow{\sim} \{d\text{-series in } \mathcal{E}(\mathbf{G}^{\mathsf{F}},1)\}.$$

This theorem suggests the following strategy to prove that $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ has only one $\sim^1$-equivalence class. If $D \subset \mathbb{N}^*$ is a finite set of non-zero integers, define a *D-series* as a subset of $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ that is a union of d-series for each $d \in D$, and which is *minimal* for this property.

2.3.2. **Lemma.** *Suppose there exists D such that*

(1) *$\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a D-series.*
(2) *Each $d \in D$ is the order of q modulo some ℓ as in Theorem 2.3.1.*

Then $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ consists of a unique $\sim^1$-equivalence class.

Proof. Use ii) to pick a prime ℓ_d satisfying the assumptions of Theorem 2.3.1 and such that q has order d modulo ℓ_d, for each $d \in D$. Then i) tells us that $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is the only non-empty subset of itself that is stable under $\sim_{\ell_d}$-equivalence for all $d \in D$. A fortiori, it is the only non-empty subset of itself that is stable under $\sim_{\ell}$-equivalence for all primes $\ell \neq p$. Hence it is a single $\sim^1$-equivalence class. □

Finding a suitable D will be done below via a case by case analysis (recall that we have reduced to the case where $\mathbf{G}$ is simple). Then, in order to find suitable primes, the following result will be useful.

2.3.3. **Theorem** ([BV04, Thm. V]). *For any $d \geqslant 3$, there exists a prime number ℓ such that q has order d modulo ℓ, with the exception of $(q,d) = (2,6)$.*

Note that if d is the order of q modulo ℓ, then $d \geqslant k \Rightarrow \ell > k$. We now proceed to the case by case analysis.

2.3.4. **Theorem.** *If $\mathbf{G}$ has type $\mathbf{A}_n$ or ${}^{2}\mathbf{A}_n$ then $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is composed of only one $\sim^1$-equivalence class.*

Proof. The case q odd is covered by Theorem 2.2.3, so let us assume q even. Since $q+1$ is odd, any prime divisor ℓ of $q+1$ is odd and good for $\mathbf{G}$, and the order of q modulo ℓ is 2. Therefore Theorem 2.3.1 tells us that the ℓ-blocks of $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ are the 2-series.

Now, for type $\mathbf{A}_n$ (split case), it is well known that $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a single 1-series, and by "Ennola duality" [BMM93, Thm 3.3], it follows that in type ${}^{2}\mathbf{A}_n$, the set

$\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a single 2-series. Therefore it is a single ℓ-block for $\ell|(q+1)$ hence also a single $\sim^1$-equivalence class.

Similarly, when $q>2$, any prime ℓ dividing $q-1$ is odd and good for $\mathbf{G}$, hence Theorem 2.3.1 tells us that the ℓ-blocks of $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ for $\ell|(q-1)$ are the 1-series. In type $\mathbf{A}_n$ and for such an ℓ, it follows that the set $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a single ℓ-block hence also a single $\sim^1$-equivalence class.

It remains to deal with the case $q=2$ in type $\mathbf{A}_n$. In this case, we will show that $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a $\{2,3\}$-series, and then we can conclude using Theorem 2.3.3 and Lemma 2.3.2. To compute $\{2,3\}$-series for $\mathbf{A}_n$, we again use Ennola duality, which asserts that they correspond bijectively to $\{1,6\}$-series for ${}^2\mathbf{A}_n$. These $\{1,6\}$-series have been computed in [Lan20, Section 3.3], to which we refer for the notion of "defect" of a 1-series. In particular, [Lan20, Prop. 3.3.9] shows that there is a unique $\{1,6\}$-series for ${}^2\mathbf{A}_n$ provided we can prove that the defect k of any 1-series of ${}^2\mathbf{A}_n$ satisfies $(k^2-3k+2)/2 \leqslant n-2$. But by [Lan20, Lem. 3.3.6], we at least have $k(k+1)/2 \leqslant n+1$. Since $(k^2-3k+2)/2 = k(k+1)/2-(2k-1)$, we thus get the desired equality if $2k-1 \geqslant 3$, that is $k \geqslant 2$. Moreover, for $k=1$, we have $(k^2-3k+2)/2=0$, thus the desired inequality also holds. □

2.3.5. **Theorem.** *If $\mathbf{G}$ has type $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ or ${}^2\mathbf{D}_n$ then $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is composed of only one $\sim^1$-equivalence class.*

Proof. Again, the case q odd is covered by Theorem 2.2.3, so we assume q is even. Picking a divisor of $q+1$ and applying Theorem 2.3.3 to $d=4$, we see thanks to Lemma 2.3.2 that it suffices to prove that $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a single $\{2,4\}$-series.

We will first exhibit a bijection between $\{2,4\}$-series and $\{1,4\}$-series, which will leave us with actually proving that $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a single $\{1,4\}$-series

To do so, we use the combinatorics of Lusztig symbols as in [Lan20, Section 3.4]. Let $\Sigma=\{S,T\}$ be a symbol ($S,T\subseteq\mathbb{N}$). We define S_e to be the subset of S composed of the even elements, that is $S_e := S\cap 2\mathbb{N}$, and S_o for the subset of odd elements, $S_o := S\cap(2\mathbb{N}+1)$. We do the same thing for T, $T=T_e\cup T_o$. Now, we define an involution φ on symbols by $\varphi(\Sigma) := \{S_e\cup T_o, T_e\cup S_o\}$. Note that $\mathrm{rank}(\Sigma)=\mathrm{rank}(\varphi(\Sigma))$ and that $\mathrm{defect}(\Sigma)$ and $\mathrm{defect}(\varphi(\Sigma))$ have the same parity. In the case of $\mathbf{D}_n$ or ${}^2\mathbf{D}_n$, when the defect is even, the congruence modulo 4 is not necessary preserved by φ. However, the congruence mod 4 of $\mathrm{defect}(\varphi(\Sigma))$ only depends on the congruence modulo 4 of $\mathrm{defect}(\Sigma)$ and $\mathrm{rank}(\Sigma)$. Indeed

$$\mathrm{defect}(\varphi(\Sigma))-\mathrm{defect}(\Sigma)=||S_e|+|T_o|-|S_o|-|T_e||-||S_e|+|S_o|-|T_e|-|T_o||.$$

Thus, the congruence modulo 4 of $\mathrm{defect}(\varphi(\Sigma))-\mathrm{defect}(\Sigma)$ depends on the parity of $|S_o|+|T_o|$ (or $|S_e|+|T_e|$). Since the defect is even, $|S|+|T|$ is even, hence $|S_o|+|T_o|\equiv|S_e|+|T_e| \pmod 2$. Now,

$$\mathrm{rank}(\Sigma)=\sum_{x\in S}x+\sum_{y\in T}y+\left[\left(\frac{|S|+|T|-1}{2}\right)^2\right],$$

thus

$$\mathrm{rank}(\Sigma)\equiv|S_o|+|T_o|+\left[\left(\frac{|S|+|T|-1}{2}\right)^2\right] \pmod 2.$$

But, $|S|+|T|$ is even, $|S|+|T|=2k$ and $[((|S|+|T|-1)/2)^2]=k(k-1)$ is even. Thus $\mathrm{defect}(\varphi(\Sigma))-\mathrm{defect}(\Sigma)$ only depends modulo 4 on the parity of $\mathrm{rank}(\Sigma)$.

Now, under the involution φ, 1-hooks correspond to 1-co-hooks and 2-co-hooks to 2-co-hooks. Hence, φ sends $\{2,4\}$-series to $\{1,4\}$-series, as claimed above.

We now prove that $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is a $\{1,4\}$-series in the different cases.

For $\mathbf{G}$ of type $\mathbf{B}_n$ or $\mathbf{C}_n$, by [Lan20, Prop. 3.4.6], we need to show that if we have a 1-series of defect k then $(k^2-4k+3)/4 \leqslant n-2$. If we have a 1-series of defect

k and rank n then $(k^2-1)/4 \leqslant n$. Since $(k^2-4k+3)/4 = (k^2-1)/4-(k-1)$, we get the desired equality if $k-1 \geqslant 2$, that is $k \geqslant 3$. For $k=1$, $(k^2-4k+3)/4 = 0$, thus the desired inequality also holds.

For $\mathbf{G}$ of type $\mathbf{D}_n$ or ${}^2\mathbf{D}_n$, by [Lan20, Prop. 3.4.6], we need to show that if we have a 1-series of defect k then $(k^2-4k+4)/4 \leqslant n-2$. If we have a 1-series of defect k and rank n then $k^2/4 \leqslant n$. Since $(k^2-4k+4)/4 = k^2/4-(k-1)$, we get the desired equality if $k-1 \geqslant 2$, that is $k \geqslant 3$. For $k=2$, $(k^2-4k+3)/4 = 0$, thus the desired inequality still holds. □

2.3.6. **Theorem.** *If $\mathbf{G}$ has type $\mathbf{F}_4$, ${}^3\mathbf{D}_4$, $\mathbf{G}_2$, $\mathbf{E}_6$, ${}^2\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$ then $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$ is composed of only one $\sim^1$-equivalence class.*

Proof. First, let us begin by $\mathbf{G}$ of type ${}^3\mathbf{D}_4$, $\mathbf{E}_6$, ${}^2\mathbf{E}_6$ or $\mathbf{E}_7$. To use Theorem 2.3.1, we need to have $\ell \geqslant 5$. Thus by Theorem 2.3.3 we can use every $d \geqslant 3$ and $d \neq 6$. Looking at the list of unipotent characters in [Car93, §13.9] and tables of d-series in [BMM93], we see that we may apply Lemma 2.3.2 with the following sets: $D = \{3,12\}$ for ${}^3\mathbf{D}_4$, $D = \{3,4\}$ for $\mathbf{E}_6$, $D = \{3,4,12,18\}$ for ${}^2\mathbf{E}_6$ and $D = \{3,4,14\}$ for $\mathbf{E}_7$.

For $\mathbf{E}_8$, we need $\ell \geqslant 7$ to use Theorem 2.3.1. We will again conclude with Lemma 2.3.2 by looking at tables. If $q \neq 2$, we can take $d \geqslant 5$ by Theorem 2.3.3 and $D = \{5,6,7,8,10,30\}$ works. If $q=2$, we can take this time $d \geqslant 3$ $d \neq 4,6$ (since the order of 2 modulo 7 is 3) and we choose $D = \{3,5,8,10,15\}$.

For $\mathbf{F}_4$, the same methods apply for $q \neq 2$ with $D = \{3,4,6,12\}$ (we can choose any $d \geqslant 3$). When $q=2$, Lemma 2.3.2 is not enough to conclude. The set of d such that q is of order d modulo some $\ell \geqslant 3$ is $d \geqslant 3$ and $d \neq 6$ by Theorem 2.3.3. However, the two unipotent characters $\phi'_{9,6}$ and $\phi''_{9,6}$ (with the notations of [Car93, Section 13.9]) are d-cuspidal for every $d \geqslant 3$, $d \neq 6$. The rest of the unipotent characters $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1) \setminus \{\phi'_{9,6}, \phi''_{9,6}\}$ form a $\{3,4,8,12\}$-series. To deal with $\phi'_{9,6}$ and $\phi''_{9,6}$, we take $\ell = 3$, and they are in the principal 3-block of $\mathbf{F}_4(2)$ by [His97].

We are left with $\mathbf{G}_2$. If q is odd, we can take $d \geqslant 3$. The two unipotent characters $\phi'_{1,3}$ and $\phi''_{1,3}$ are d-cuspidal for every $d \geqslant 3$ and $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1) \setminus \{\phi'_{1,3}, \phi''_{1,3}\}$ is a $\{3,6\}$-series. We take $\ell = 2$ and the tables in [HS92] give us that $\phi'_{1,3}$ and $\phi''_{1,3}$ are in the principal 2-block. This concludes the case q odd. Now, if q is even and $q \neq 2$, we can still use $d \geqslant 3$. Thus, we have the same issue with $\phi'_{1,3}$ and $\phi''_{1,3}$. We can no longer use $\ell = 2$ but we can use $\ell = 3$, and the tables in [HS90] gives us that $\phi'_{1,3}$ and $\phi''_{1,3}$ are in the principal 3-block. Finally, if $q = 2$, we can now use $d \geqslant 3$ and $d \neq 6$. All the unipotent characters are in the principal 3-series apart from $\phi'_{1,3}$, $\phi''_{1,3}$, $\phi_{2,1}$ and $\mathbf{G}_2[-1]$. But we see in [HS90] that they all are in the principal 3-block.

□

We have now completed the proof of Theorem 2.0.1 for all reductive $(\mathbf{G},\mathsf{F})$. Indeed, by Proposition 2.1.2, the statement of this theorem is equivalent to the statement that there is only one $\sim$-equivalence class in $\mathrm{Irr}(\mathbf{G}^{\mathsf{F}})$. Then Proposition 2.2.1 shows it is enough to prove that there is only one $\sim^1$-equivalence class in $\mathcal{E}(\mathbf{G}^{\mathsf{F}},1)$, and Proposition 2.2.2 reduces to the case of simple (and adjoint) $\mathbf{G}$. All the simple cases are then covered by Theorems 2.2.3, 2.3.4, 2.3.5 and 2.3.6.

3. p-ADIC GROUPS

Here $\mathbf{G}$ is a reductive group over a local non-archimedean field F with residue field $k_F := \mathbb{F}_q$. We put $G := \mathbf{G}(F)$. For any commutative ring R in which p is invertible, we denote by $\mathrm{Rep}_R(G)$ the category of smooth RG-modules. The Bernstein center $\mathfrak{Z}_R(G)$ is by definition the center of this category.

3.1. **The depth 0 summand.** Denote by $\mathcal{B}$ the (reduced) Bruhat-Tits building associated to $\mathbf{G}$. This is a polysimplicial complex equipped with a polysimplicial action of G. To any facet σ of $\mathcal{B}$ is associated a parahoric subgroup G_σ, which is open, compact and contained in the pointwise stabilizer of σ. It is the group $\mathbf{G}_\sigma(\mathcal{O}_F)$ of $\mathcal{O}_F$-valued points of a certain smooth $\mathcal{O}_F$-model $\mathbf{G}_\sigma$ of $\mathbf{G}$. We denote by $\overline{\mathbf{G}}_\sigma$ the reductive quotient of the special fiber of $\mathbf{G}_\sigma$. Then $\overline{G}_\sigma := \overline{\mathbf{G}}_\sigma(\mathbb{F}_q)$ is also the quotient of G_σ by its pro-p-radical G_σ^+. Since G_σ^+ is open and pro-p, there is an averaging idempotent $e_\sigma^+ \in \mathcal{H}_R(G_\sigma) \subset \mathcal{H}_R(G)$ in the Hecke algebra $\mathcal{H}_R(G)$ of G with coefficients R. Recall that $\mathcal{H}_R(G)$ acts on any smooth RG-module V.

3.1.1. **Definition.** A smooth RG-module V has *depth 0* if $V = \sum_{x \in \mathcal{B}_0} e_x^+ V$.

Here $\mathcal{B}_0$ is the set of vertices of $\mathcal{B}$. It is known [Dat09, Appendice A] that the full subcategory $\mathrm{Rep}_R^0(G)$ of $\mathrm{Rep}_R(G)$ composed of depth 0 objects is a direct factor abelian subcategory. Correspondingly, there is an idempotent $\varepsilon_0 \in \mathfrak{Z}_R(G)$ that projects any object V onto its depth 0 factor. When $R = \mathbb{C}$, the Bernstein decomposition of $\mathrm{Rep}_\mathbb{C}^0(G)$ as a sum of blocks was made explicit by Morris in [Mor99].

3.2. **(un)refined depth 0 types.** Following Moy and Prasad, we define an *unrefined depth* 0 *type* to be a pair (σ, π), where $\sigma \subset \mathcal{B}$ is a facet and π is an irreducible complex cuspidal representation of $\overline{G}_\sigma$. We also denote by π the inflation of π to G_σ. Then [Mor99, Theorem 4.8] tells us that $\mathrm{ind}_{G_\sigma}^G(\pi)$ is a projective generator for a certain sum of Bernstein components of depth 0. Obviously, this representation only depends on the G-conjugacy class $\mathfrak{t}$ of (σ, π), whence a direct factor $\mathrm{Rep}_\mathbb{C}^\mathfrak{t}(G)$ of $\mathrm{Rep}_\mathbb{C}^0(G)$. Denote by $\mathfrak{T}$ the set of G-conjugacy classes of such pairs. It turns out that, for $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}$, the factors $\mathrm{Rep}_\mathbb{C}^\mathfrak{t}(G)$ and $\mathrm{Rep}_\mathbb{C}^{\mathfrak{t}'}(G)$ are either orthogonal or equal. Whence an equivalence relation $\sim$ on $\mathfrak{T}$ and a decomposition

$$\mathrm{Rep}_\mathbb{C}^0(G) = \prod_{[\mathfrak{t}] \in \mathfrak{T}/\sim} \mathrm{Rep}_\mathbb{C}^{[\mathfrak{t}]}(G).$$

We refer to [Lan20, Section 2.2] for an explicit description of the relation $\sim$.

In general, the factors $\mathrm{Rep}_\mathbb{C}^{[\mathfrak{t}]}(G)$ are further decomposable. Suppose $(\sigma, \pi) \in [\mathfrak{t}]$ and denote by $G_\sigma^\dagger$ the maximal compact subgroup of the stabilizer of σ in G. By [Mor99, Theorem 4.9], for any irreducible subquotient $\pi^\dagger$ of $\mathrm{ind}_{G_\sigma}^{G_\sigma^\dagger}(\pi)$, the representation $\mathrm{ind}_{G_\sigma^\dagger}^G(\pi^\dagger)$ is the projective generator of a single Bernstein block. Also, any block inside $\mathrm{Rep}_\mathbb{C}^{[\mathfrak{t}]}(G)$ is obtained in this way. For later reference, let us notice that, since $G_\sigma^\dagger/G_\sigma$ is abelian, any other irreducible subquotient of $\mathrm{ind}_{G_\sigma}^{G_\sigma^\dagger}(\pi)$ is a twist of $\pi^\dagger$ by a character of $G_\sigma^\dagger/G_\sigma$.

3.3. **Systems of idempotents.** Fix $\mathfrak{t} \in \mathfrak{T}$, and let $[\mathfrak{t}]$ denote its equivalence class. For a facet $\tau \in \mathcal{B}$, define $e_{[\mathfrak{t}],\tau} \in \mathcal{H}_R(G_\tau)$ to be the idempotent that cuts out all irreducible representations of $\overline{G}_\tau$ whose cuspidal support contains $(\overline{G}_\sigma, \pi)$ for some facet σ containing τ and π such that $(\sigma, \pi) \in [\mathfrak{t}]$. Then the system $(e_{[\mathfrak{t}],\tau})_{\tau \in \mathcal{B}_\bullet}$ is 0-*consistent* in the sense of [Lan20, Def. 2.1.3], that is :

(1) $\forall x \in \mathcal{B}_0, \forall g \in G,\ e_{[\mathfrak{t}],gx} = g e_{[\mathfrak{t}],x} g^{-1}$
(2) $\forall \tau \in \mathcal{B}_\bullet, \forall x \in \mathcal{B}_0,\ x \in \bar{\tau} \Rightarrow e_{[\mathfrak{t}],\tau} = e_\tau^+ e_{[\mathfrak{t}],x}$.

Moreover, Proposition 2.2.3 of [Lan20] implies that

$$\mathrm{Rep}_\mathbb{C}^{[\mathfrak{t}]}(G) = \left\{ V \in \mathrm{Rep}_\mathbb{C}(G),\ V = \sum_{x \in \mathcal{B}_0} e_{[\mathfrak{t}],x} V \right\}.$$

Denote by N the l.c.m of all $|\overline{G}_\tau|$, $\tau\in\mathcal{B}_\bullet$. Since each $e_{[\mathfrak{t}],\tau}$ lies in $\mathcal{H}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)$, we see that the summand $\mathrm{Rep}^{[\mathfrak{t}]}_{\mathbb{C}}(G)$ is "defined over $\overline{\mathbb{Z}}[\frac{1}{N}]$" in the sense that the corresponding idempotent $\varepsilon_{[\mathfrak{t}]}$ of $\mathfrak{Z}_{\mathbb{C}}(G)$ lies in $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)$, and we have a decomposition

$$\mathrm{Rep}^0_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)=\prod_{[\mathfrak{t}]\in\mathfrak{T}/\sim}\mathrm{Rep}^{[\mathfrak{t}]}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G).$$

Now, to a subset T of $\mathfrak{T}/\sim$ we associate an idempotent $\varepsilon_T:=\sum_{[\mathfrak{t}]\in T}\varepsilon_{[\mathfrak{t}]}\in\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)$ in the Bernstein center, and a consistent system of idempotents $(e_{T,\tau})_{\tau\in\mathcal{B}_\bullet}$ given by $e_{T,\tau}:=\sum_{[\mathfrak{t}]\in T}e_{[\mathfrak{t}],\tau}$ for any facet $\tau\subset\mathcal{B}$. The following observation is crucial for the argument.

3.3.1. **Proposition.** *We have $\varepsilon_T\in\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)\Leftrightarrow\forall\tau\in\mathcal{B}_\bullet$, $e_{T,\tau}\in\mathcal{H}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G_\tau)$.*

Proof. As recalled above, the direct factor category associated to the idempotent ε_T is given by

$$\varepsilon_T\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)=\left\{V\in\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G),\,V=\sum_{x\in\mathcal{B}_0}e_{T,x}V\right\}.$$

Moreover, its orthogonal complement in $\mathrm{Rep}^0_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)$ is the category similarly associated to the complement subset T^c of T in $\mathfrak{T}/\sim$. Therefore, if all $e_{T,\tau}$ are $\overline{\mathbb{Z}}[\frac{1}{p}]$-valued, then so are all $e_{T^c,\tau}=e^+_\tau-e_{T,\tau}$, so the decomposition $\mathrm{Rep}^0_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)=\varepsilon_T\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)\times\varepsilon_{T^c}\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{N}]}(G)$ is defined over $\overline{\mathbb{Z}}[\frac{1}{p}]$, hence the idempotents ε_T and ε_{T^c} belong to $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. Conversely, if $\varepsilon_T\in\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$, then the equality $e_{T,\tau}=\varepsilon_T*e^+_\tau$ in $\mathcal{H}_{\mathbb{C}}(G)$ from the proof of Lemma 4.1.2 in [Lan20], shows that $e_{T,\tau}\in\mathcal{H}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. □

3.3.2. **Corollary.** *We have $\varepsilon_T\in\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ if and only if $T=\mathfrak{T}/\sim$, i.e. if and only if $\varepsilon_T=\varepsilon_0$.*

Proof. Suppose $\varepsilon_T\in\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. By the last proposition, for any facet τ in $\mathcal{B}$, the idempotent $e_{T,\tau}$ is inflated from a central idempotent of $\overline{\mathbb{Z}}[\frac{1}{p}]\overline{G}_\tau$. By Theorem 2.0.1, there is no non-trivial central idempotent in $\overline{\mathbb{Z}}[\frac{1}{p}]\overline{G}_\tau$. So $e_{T,\tau}$ is either e^+_τ or 0. There is certainly a vertex x such that $e_{T,x}\neq 0$, and thus $e_{T,x}=e^+_x$. Then, for a chamber σ containing x, we have $e_{T,\sigma}=e^+_\sigma e_{T,x}=e^+_\sigma$. By G-equivariance it follows that $e_{T,\tau}=e^+_\tau$ for all chambers, which implies $e_{T,\tau}\neq 0$ and therefore $e_{T,\tau}=e^+_\tau$ for all facets. Hence $\varepsilon_T=\varepsilon_0$ and $T=\mathfrak{T}/\sim$. □

When G is semi-simple and simply-connected, we have $G_\tau=G^\dagger_\tau$ for all facets τ so, from our discussion of depth 0 types above, each $\mathrm{Rep}^{[\mathfrak{t}]}_{\mathbb{C}}(G)$ is already a block of $\mathrm{Rep}^0_{\mathbb{C}}(G)$ and, therefore, any idempotent of $\varepsilon_0\mathfrak{Z}_{\mathbb{C}}(G)$ is equal to some ε_T for $T\subset\mathfrak{T}/\sim$. So we have proved :

3.3.3. **Corollary.** *If G is semisimple and simply-connected, $\mathrm{Rep}^0_{\overline{\mathbb{Z}}[1/p]}(G)$ is a block (equivalently, ε_0 is primitive in $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$).*

3.3.4. *Remark.* As in the case of finite groups, this kind of results can be interpreted at the level of irreducible $\overline{\mathbb{Q}}G$-modules in the following way. For $M\in\mathbb{N}^*$ and $\pi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$, denote by $\varepsilon_{M,\pi}$ the unique primitive idempotent of $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{M}]}(G)$ that acts as identity on π. So, if $M=N$ (defined as above), $\varepsilon_{N,\pi}$ is also primitive in $\mathfrak{Z}_{\overline{\mathbb{Q}}}(G)$ and defines the Bernstein component that contains π. Let $\ell\neq p$ be a prime and denote by $N_{\ell'}$ the prime-to-ℓ part of N. Then, for $\pi,\pi'\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$, the following properties are equivalent :

(1) $\varepsilon_{N_{\ell'},\pi} = \varepsilon_{N_{\ell'},\pi'}$
(2) For any embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, the base changed representations π, π' belong to the same block of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G)$.

This justifies calling the equivalence classes for the relation $\pi \sim_\ell \pi' \Leftrightarrow \varepsilon_{N_{\ell'},\pi} = \varepsilon_{N_{\ell'},\pi'}$ the *ℓ-blocks of* $\mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$. They correspond to minimal "ℓ-integral subsets" of the set of Bernstein blocks. Similarly, the blocks of $\mathrm{Rep}_{\overline{\mathbb{Z}}(\frac{1}{p}]}(G)$ correspond to minimal subsets of the set of Bernstein blocks that are ℓ-integral for all $\ell \neq p$. In other words, the equivalence relation $\sim$ generated on $\mathrm{Irr}_{\overline{\mathbb{Q}}}(G)$ by all $\sim_\ell$, $\ell \neq p$ satisfies $\pi \sim \pi' \Leftrightarrow \pi$ and π' belong to the same block of $\mathrm{Rep}^0_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$.

Now, to tackle the general case, we need to recall some facts about the quotients $G_\tau^\dagger/G_\tau$.

3.4. **The Kottwitz map.** We first recall a definition of Borovoi :

$$\pi_1(\mathbf{G}) := X_*(\mathbf{T})/\langle\Phi^\vee\rangle = \mathrm{coker}(X_*(\mathbf{T}_{\mathrm{sc}}) \longrightarrow X_*(\mathbf{T})).$$

Here $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, $\Phi^\vee \subset X_*(\mathbf{T})$ is the set of (absolute) coroots, and $\mathbf{T}_{\mathrm{sc}}$ is the inverse image of $\mathbf{T}$ in the simply connected covering $\mathbf{G}_{\mathrm{sc}}$ of the derived group $\mathbf{G}_{\mathrm{der}}$ of $\mathbf{G}$. Using the fact that all tori are conjugate, the group $\pi_1(\mathbf{G})$ turns out to be canonically independent of the choice of $\mathbf{T}$. Moreover, if $\mathbf{T}$ is chosen so as to be defined over F, then $\pi_1(\mathbf{G})$ gets a $\mathbb{Z}$-linear action of the Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$. Again, this action does not depend on the choice of F-rational torus $\mathbf{T}$ (although two such choices may not be $\mathbf{G}(F)$-conjugate).

Now, let $I_F \subset \Gamma_F$ denote the inertia subgroup and let F denote the geometric Frobenius in Γ_F/I_F. Kottwitz has defined a surjective morphism

$$\kappa_G : \ G = \mathbf{G}(F) \longrightarrow \pi_1(G) := (\pi_1(\mathbf{G})_{I_F})^{\mathsf{F}}.$$

We refer to [KP22, Chap. 12] for the detailed construction of this map. The following properties of this map are particularly relevant to our problem :

- The kernel $G^0 := \ker\kappa_G$ is the subgroup of G generated by parahoric subgroups and, for any facet $\sigma \subset \mathcal{B}$, the parahoric group G_σ is the stabilizer of σ in G^0.
- The inverse image $G^1 := \kappa_G^{-1}(\pi_1(G)_{\mathrm{tors}})$ is the subgroup of G generated by compact subgroups and, for any facet $\sigma \subset \mathcal{B}$, the compact open group $G_\sigma^\dagger$ introduced above is the stabilizer of σ in G^1.

In particular, we have $G_\sigma^\dagger \subset G^1$ and $G_\sigma = G^0 \cap G_\sigma^\dagger$.

Now let Ψ_G be the diagonalizable algebraic group scheme over $\overline{\mathbb{Z}}[\frac{1}{p}]$ associated to the finitely generated abelian group $\pi_1(G)$. Its maximal subtorus Ψ_G^t is the usual "torus of unramified characters" of G, while the quotient $\Psi_G^f := \Psi_G/\Psi_G^t$ is the diagonalizable group scheme associated to the finite group $\pi_1(G)_{\mathrm{tors}}$.

For any $\mathbb{Z}[\frac{1}{p}]$-algebra R, the group $\Psi_G(R) = \mathrm{Hom}(\pi_1(G), R^\times)$ identifies via κ_G to a group of R-valued characters of G, hence it acts on the category $\mathrm{Rep}_R(G)$ by twisting the representations. Since this action is R-linear, this induces in turn an action of $\Psi_G(R)$ by automorphisms of R-algebra on $\mathfrak{Z}_R(G)$, hence an action on the set $\mathrm{Idemp}(\mathfrak{Z}_R(G))$ of idempotents of $\mathfrak{Z}_R(G)$.

The idempotents of $\mathfrak{Z}_\mathbb{C}(G)$ are known to be supported on the set of compact elements hence in particular on G^1, so the action of $\Psi_G(\mathbb{C})$ on $\mathrm{Idemp}(\mathfrak{Z}_\mathbb{C}(G))$ factors through an action of $\Psi_G^f(\mathbb{C}) = \mathrm{Hom}(\pi_1(G)_{\mathrm{tors}}, \mathbb{C}^\times) = \pi_0(\Psi_{G,\mathbb{C}})$. Further, the depth 0 projector ε_0 is known to be supported on the set of topologically unipotent elements, hence in particular on G^0, so ε_0 is invariant under the action of $\Psi_G^f(\mathbb{C})$.

3.4.1. **Lemma.** *For each* $[\mathfrak{t}] \in \mathfrak{T}/\sim$*, the associated idempotent* $\varepsilon_{[\mathfrak{t}]}$ *in* $\mathfrak{Z}_{\mathbb{C}}(G)$ *is invariant by* $\Psi_G^f(\mathbb{C})$*. Moreover, the primitive idempotents that refine* $\varepsilon_{[\mathfrak{t}]}$ *form a single* $\Psi_G^f(\mathbb{C})$*-orbit.*

Proof. Pick $(\sigma, \pi) \in [\mathfrak{t}]$. We know that the direct factor category $\mathrm{Rep}_{\mathbb{C}}^{[\mathfrak{t}]}(G)$ is generated by the projective object $\mathrm{ind}_{G_\sigma}^G(\pi)$. For any $\psi \in \Psi_G(\mathbb{C})$, we have $\mathrm{ind}_{G_\sigma}^G(\pi)\otimes\psi = \mathrm{ind}_{G_\sigma}^G(\pi \otimes \psi_{|G_\sigma}) = \mathrm{ind}_{G_\sigma}^G(\pi)$. It follows that $\mathrm{Rep}_{\mathbb{C}}^{[\mathfrak{t}]}(G)$ is stable under the action of $\Psi_G(\mathbb{C})$, hence $\varepsilon_{[\mathfrak{t}]}$ is invariant.

Now, we also know that any block of $\mathrm{Rep}_{\mathbb{C}}^{[\mathfrak{t}]}(G)$ is generated by a projective object of the form $\mathrm{ind}_{G_\sigma^\dagger}^G(\pi^\dagger)$ where $\pi^\dagger$ is an irreducible constituent of $\mathrm{ind}_{G_\sigma}^{G_\sigma^\dagger}(\pi)$. But any two such irreducible constituents are twists of one another by a character of $G_\sigma^\dagger/G_\sigma$. Moreover, the latter group embeds in $\pi_1(G)_{\mathrm{tors}}$ via κ_G, hence the restriction map $\Psi_G^f(\mathbb{C}) \longrightarrow \mathrm{Hom}(G_\sigma^\dagger/G_\sigma, \mathbb{C}^\times)$ is surjective, and the second statement of the lemma follows. □

This lemma implies that the only central idempotents of depth 0 in $\mathfrak{Z}_{\mathbb{C}}(G)$ that are invariant by $\Psi_G^f(\mathbb{C})$ are the ε_T for $T \subset \mathfrak{T}/\sim$. Now, observe that the action of $\Psi_G(\overline{\mathbb{Z}}[\frac{1}{p}])$ on $\mathrm{Idemp}(\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G))$ factors over $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])$, which is equal to $\Psi_G^f(\mathbb{C})$. In other words, $\Psi_G^f(\mathbb{C})$ preserves the subset of idempotents of $\mathfrak{Z}_{\mathbb{C}}(G)$ that belong to $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. Therefore, we can restate Corollary 3.3.2 as follows :

3.4.2. **Corollary.** *The only* $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])$*-invariant idempotent of* $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ *is* ε_0*. Hence* $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])$ *acts transitively on the set of primitive idempotents of* $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$*.*

In order to better understand the action of $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])$ on $\mathrm{Idemp}(\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G))$, write this group as a product $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}]) = \Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])_p \times \Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])_{p'}$ of a p-group and a p'-group.

3.4.3. **Lemma.** *Any idempotent of* $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ *is invariant under* $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])_{p'}$*.*

Proof. Let ε be an idempotent of $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ and assume, without loss of generality, that it is primitive. On the other hand, let $\psi \in \Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])_{p'}$ and assume, without loss of generality that ψ has order a power of some prime $\ell \neq p$. In order to prove that $\psi \cdot \varepsilon = \varepsilon$, it suffices to find a non-zero object (V, π) in $\varepsilon\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ such that $\pi \otimes \psi \simeq \pi$. Since the composition of $\psi : \pi_1(G)_{\mathrm{tors}} \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$ with any morphism $\overline{\mathbb{Z}}[\frac{1}{p}] \longrightarrow \overline{\mathbb{F}}_\ell$ is trivial, it suffices to find a non-zero object (V, π) in $\varepsilon\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ whose $\overline{\mathbb{Z}}[\frac{1}{p}]$-module structure factors over a morphism $\overline{\mathbb{Z}}[\frac{1}{p}] \longrightarrow \overline{\mathbb{F}}_\ell$. But $\varepsilon\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ certainly contains a representation of the form $\varepsilon.\mathrm{ind}_H^G(\overline{\mathbb{Z}}[\frac{1}{p}])$ for some open pro-p-subgroup H of G. Such a representation being projective as a $\overline{\mathbb{Z}}[\frac{1}{p}]$-module, its reduction modulo any maximal ideal containing ℓ is non-zero and satisfies the desired property. □

In the next statement, we say that a torus $\mathbf{T}$ defined over F is *P_F-induced* if the action of the wild inertia subgroup P_F permutes a basis of $X_*(\mathbf{T})$.

3.4.4. **Corollary.** *Suppose that p does not divide* $|\pi_1(\mathbf{G}_{\mathrm{der}})|$ *and that the torus* $\mathbf{G}_{\mathrm{ab}}$ *is P_F-induced. Then* ε_0 *is a primitive idempotent of* $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$*.*

Proof. From the inclusions $X_*(\mathbf{T}_{\mathrm{sc}}) \subset X_*(\mathbf{T}_{\mathrm{der}}) \subset X_*(\mathbf{T})$ and the isomorphism $X_*(\mathbf{T})/X_*(\mathbf{T}_{\mathrm{der}}) \xrightarrow{\sim} X_*(\mathbf{G}_{\mathrm{ab}})$, we get an exact sequence $\pi_1(\mathbf{G}_{\mathrm{der}}) \hookrightarrow \pi_1(\mathbf{G}) \twoheadrightarrow$

$X_*(\mathbf{G}_{\mathrm{ab}})$. Applying I_F-coinvariants, we get an exact sequence $\pi_1(\mathbf{G}_{\mathrm{der}})_{I_F} \longrightarrow \pi_1(\mathbf{G})_{I_F} \twoheadrightarrow X_*(\mathbf{G}_{\mathrm{ab}})_{I_F}$. Since $\pi_1(\mathbf{G}_{\mathrm{der}})$ is finite, this sequence remains exact on the torsion subgroups and the p-torsion subgroups, so we get an exact sequence

$$\pi_1(\mathbf{G}_{\mathrm{der}})_{I_F,p-\mathrm{tors}} \longrightarrow \pi_1(\mathbf{G})_{I_F,p-\mathrm{tors}} \twoheadrightarrow X_*(\mathbf{G}_{\mathrm{ab}})_{I_F,p-\mathrm{tors}}.$$

Our first assumption implies that $\pi_1(\mathbf{G}_{\mathrm{der}})_{I_F,p-\mathrm{tors}} = 0$, and our second assumption implies that $X_*(\mathbf{G}_{\mathrm{ab}})_{I_F,p-\mathrm{tors}} = (X_*(\mathbf{G}_{\mathrm{ab}})_{P_F,\mathrm{tors}})_{I_F} = 0$. Therefore $\pi_1(G)_{p-\mathrm{tors}} = \{0\}$, thus $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])_p = \{1\}$ and we conclude thanks to Lemma 3.4.3 and Corollary 3.4.2. □

3.5. **Special points.** Let $\mathbf{S}$ be a maximal split torus in $\mathbf{G}$ and let $\mathbf{Z}$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$. This is a Levi component of a minimal F-rational parabolic subgroup of $\mathbf{G}$. By [KP22, Lemma 12.5.6], we know that the canonical map $\pi_1(Z) \longrightarrow \pi_1(G)$ is injective on torsion subgroups. Correspondingly, the map $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}]) \longrightarrow \Psi_Z^f(\overline{\mathbb{Z}}[\frac{1}{p}])$ is surjective. Note that the quotient of $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])$ thus obtained is independent of the choice of $\mathbf{S}$ since all maximal split tori are G-conjugate.

3.5.1. **Lemma.** *The action of $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])$ on the set of idempotents of $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ factors over the quotient $\Psi_Z^f(\overline{\mathbb{Z}}[\frac{1}{p}])$.*

Proof. Let $\mathbf{S}$ and $\mathbf{Z}$ be as above, and pick a special vertex x in the apartment corresponding to $\mathbf{S}$ in $\mathcal{B}$. By [KP22, Prop. 7.7.10] we have $G_x^\dagger = G_x Z^1$, hence $G_x^\dagger/G_x = Z^1/Z^0 = \pi_1(Z)_{\mathrm{tors}}$. Therefore, if we pick a supercuspidal representation π of $\overline{G}_x$, and an irreducible subquotient $\pi^\dagger$ of $\mathrm{ind}_{G_x}^{G_x^\dagger}(\pi)$, then the corresponding primitive central idempotent $\varepsilon_{(x,\pi^\dagger)} \in \mathfrak{Z}_{\mathbb{C}}(G)$ is invariant under the kernel of $\Psi_G^f(\mathbb{C}) \longrightarrow \Psi_Z^f(\mathbb{C})$. So let ε be the unique primitive idempotent of $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ such that $\varepsilon.\varepsilon_{(x,\pi^\dagger)} = \varepsilon_{(x,\pi^\dagger)}$. By uniqueness, ε is also invariant under the kernel of $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}]) \longrightarrow \Psi_Z^f(\overline{\mathbb{Z}}[\frac{1}{p}])$. Since $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}])$ acts transitively on the set of primitive idempotents of $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$, we conclude that this action factors over $\Psi_Z^f(\overline{\mathbb{Z}}[\frac{1}{p}])$. □

3.6. **The quasi-split case : group side.** In this subsection, we assume that $\mathbf{G}$ is quasi-split over F. In this case, the centralizer $\mathbf{Z}$ of a maximal split torus $\mathbf{S}$ is itself a torus, that we denote by $\mathbf{T} := \mathbf{Z}$. According to Lemma 3.5.1, Lemma 3.4.3 and Corollary 3.4.2, the natural action of $\Psi_G^f(\overline{\mathbb{Z}}[\frac{1}{p}]) = \mathrm{Hom}(\pi_1(G)_{\mathrm{tors}}, \overline{\mathbb{Z}}[\frac{1}{p}]^\times)$ induces a transitive action of $\Psi_T^f(\overline{\mathbb{Z}}[\frac{1}{p}])_p = \mathrm{Hom}(\pi_1(T)_{p-\mathrm{tors}}, \overline{\mathbb{Z}}[\frac{1}{p}]^\times)$ on the set of primitive idempotents of $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. Therefore, if $\mathbf{T}$ is P_F-induced, we have $\pi_1(T)_{p-\mathrm{tors}} = 1$, so it follows that ε_0 is a primitive idempotent in $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. In particular we have proven the following result.

3.6.1. **Theorem.** *Suppose that $\mathbf{G}$ is quasi-split and tamely ramified over F. Then ε_0 is a primitive idempotent of $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$.*

This theorem mirrors the fact that the space of tamely ramified Langlands parameters for G is connected over $\overline{\mathbb{Z}}[\frac{1}{p}]$, under the same hypothesis, as proved in [DHKM20, Theorem 4.29]. Below we will prove more generally that for any quasi-split $\mathbf{G}$, there is a natural bijection between connected components of the space of tamely ramified Langlands parameters for G and the set of primitive idempotents in $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. On the $\mathbf{G}$-side, the main result is the following one.

3.6.2. **Theorem.** *Suppose $\mathbf{G}$ is quasi-split. Then the action of $\Psi_T^f(\overline{\mathbb{Z}}[\frac{1}{p}])_p$ on the set of primitive idempotents of $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ is simply transitive.*

Proof. Let $G' := \kappa_G^{-1}(\pi_1(T)_{p-\mathrm{tors}})$ be the inverse image in G of $\pi_1(T)_{p-\mathrm{tors}}$ by κ_G. As already mentioned, it does not depend on the choice of $\mathbf{S}$. For any facet σ in $\mathcal{B}$, we put $G'_\sigma := G_\sigma^\dagger \cap G'$. If σ belongs to the apartment associated to $\mathbf{S}$, then $G'_\sigma = G_\sigma T'$ where $T' = T \cap G' = \kappa_T^{-1}(\pi_1(T)_{p-\mathrm{tors}})$. Since $G_\sigma \cap T = T^0$, we have a short exact sequence

$$\bar{G}_\sigma = G_\sigma/G_\sigma^+ \hookrightarrow G'_\sigma/G_\sigma^+ \twoheadrightarrow \pi_1(T)_{p-\mathrm{tors}}. \tag{3.1}$$

We claim that this sequence splits canonically and, more precisely, that there is a canonical decomposition

$$G'_\sigma/G_\sigma^+ = \left(G_\sigma/G_\sigma^+\right) \times \pi_1(T)_{p-\mathrm{tors}}. \tag{3.2}$$

To see this, recall that there are canonical smooth $\mathcal{O}_F$-models $\mathbf{G}_\sigma \subset \mathbf{G}'_\sigma$ of $\mathbf{G}$ such that

- $\mathbf{G}_\sigma(\mathcal{O}_F) = G_\sigma$ and $\mathbf{G}'_\sigma(\mathcal{O}_F) = G'_\sigma$ and $(\mathbf{G}_\sigma)_{\overline{\mathbb{F}}_q} = ((\mathbf{G}'_\sigma)_{\overline{\mathbb{F}}_q})^\circ$.
- $\mathbf{G}'_\sigma$ contains the canonical model $\mathbf{T}'$ of $\mathbf{T}$ such that $\mathbf{T}'(\mathcal{O}_F) = T'$, and $\pi_0(\mathbf{T}'_{\overline{\mathbb{F}}_q}) \xrightarrow{\sim} \pi_0((\mathbf{G}'_\sigma)_{\overline{\mathbb{F}}_q})$, while we also have $\pi_0(\mathbf{T}'_{\overline{\mathbb{F}}_q}) \xrightarrow{\sim} \pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}$.
- Denote by $\overline{\mathbf{G}'_\sigma}$ the quotient of the special fiber $(\mathbf{G}'_\sigma)_{\mathbb{F}_q}$ of $\mathbf{G}'_\sigma$ by its unipotent radical. Then the short exact sequence (3.1) is obtained by taking the $\mathbb{F}_q$-rational points of the sequence
$$\overline{\mathbf{G}_\sigma} \hookrightarrow \overline{\mathbf{G}'_\sigma} \twoheadrightarrow \pi_0((\mathbf{G}'_\sigma)_{\overline{\mathbb{F}}_q}) = \pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}.$$
- Denote by $\overline{\mathbf{T}'}$ the quotient of the special fiber of $\mathbf{T}'$ by its unipotent radical. Then $\mathbf{T}' \hookrightarrow \mathbf{G}'_\sigma$ induces a closed immersion $\overline{\mathbf{T}'} \hookrightarrow \overline{\mathbf{G}'_\sigma}$ and $\overline{\mathbf{T}'} \cap \overline{\mathbf{G}_\sigma} = \overline{\mathbf{T}'}^\circ$ is a maximal torus of $\overline{\mathbf{G}_\sigma}$ and we have an exact sequence
$$\overline{\mathbf{T}'}^\circ = \overline{\mathbf{T}'} \cap \overline{\mathbf{G}_\sigma} \hookrightarrow \overline{\mathbf{T}'} \twoheadrightarrow \pi_0((\mathbf{T}')_{\overline{\mathbb{F}}_q}) = \pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}.$$

Now, since $\overline{\mathbf{T}'}^\circ(\overline{\mathbb{F}}_p)$ is a p'-torsion abelian group, $H^1(\pi_1(\mathbf{T})_{p-\mathrm{tors}}, \overline{\mathbf{T}'}^\circ(\overline{\mathbb{F}}_p)) = \{1\}$ so there exists a splitting $\iota : \pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}} \hookrightarrow \overline{\mathbf{T}'}(\overline{\mathbb{F}}_p)$ of the last exact sequence. This ι also provides a splitting $\pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}} \hookrightarrow \overline{\mathbf{G}'_\sigma}(\overline{\mathbb{F}}_p)$ of the short exact sequence in item 3 above. But since $\mathbf{T}'$ is an abelian group scheme, we see that the conjugation action of $\pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}$ on $\overline{\mathbf{G}_\sigma}$ through ι fixes pointwise the maximal torus $\overline{\mathbf{T}'}^\circ$ of $\overline{\mathbf{G}_\sigma}$. It follows that this action is inner and, more precisely, given by a morphism from $\pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}$ to the image of $\overline{\mathbf{T}'}^\circ$ in the adjoint group of $\overline{\mathbf{G}_\sigma}$. But such a morphism has to be trivial since $\pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}$ is a p-group. Hence the action of $\pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}$ through ι is trivial on $\overline{\mathbf{G}_\sigma}$ and we get a decomposition $\overline{\mathbf{G}'_\sigma} = \overline{\mathbf{G}_\sigma} \times \pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}$. Moreover, such a decomposition is unique because $\mathrm{Hom}(\pi_1(\mathbf{T})_{I_F,p-\mathrm{tors}}, Z(\overline{\mathbf{G}_\sigma}(\overline{\mathbb{F}}_p)) = \{1\}$. Taking $\mathbb{F}_q$-rational points, we get the claimed canonical decomposition (3.2).

Now, this decomposition (3.2) implies that the pro-p-radical ${G'_\sigma}^+$ of G'_σ surjects onto $\pi_1(T)_{p-\mathrm{tors}}$. For any character ψ of $\pi_1(T)_{p-\mathrm{tors}}$, we therefore get a central idempotent $e_\sigma^\psi \in \mathcal{H}_{\mathbb{Z}[\frac{1}{p}]}(G'_\sigma)$ supported on ${G'_\sigma}^+$, and we have $e_\sigma^+ = \sum_\psi e_\sigma^\psi$. Again, these idempotents do not depend on the choice of apartment containing σ, since they are given by the restriction of a global character of G' to $(G'_\sigma)^+$. In particular, they are invariant under the action of G, in the sense that $e_{g\sigma}^\psi = g e_\sigma^\psi g^{-1}$ for all $g \in G$. Moreover, if x is a vertex of the facet σ, the pro-p-radical ${G'_x}^+$ of G'_x is a normal subgroup of ${G'_\sigma}^+$ and we have ${G'_\sigma}^+ = {G'_x}^+ G_\sigma^+$. In terms of idempotents, it follows that $e_\sigma^\psi = e_\sigma^+ e_x^\psi$ for all ψ. But then, the proof of [Lan18, Prop. 1.0.6] shows that the system of idempotents $(e_\sigma^\psi)_{\sigma \in \mathcal{B}_\bullet}$ is consistent in the sense of [MS10, Def. 2.1] (note that in [Lan18, Prop. 1.0.6] the idempotents are assumed to be supported on the parahoric subgroups while here we allow support on a slightly bigger subgroup, but this is harmless for the argument there). Then, [MS10, Thm. 3.1] tells us that the full subcategories $\mathrm{Rep}^{\psi}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G) := \{V \in \mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G),\, V = \sum_{x\in\mathcal{B}_0} e^{\psi}_x V\}$ are Serre subcategories of $\mathrm{Rep}^{0}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. Since for all σ, ψ and ψ' we have $\psi \neq \psi' \Rightarrow e^{\psi}_{\sigma} e^{\psi'}_{\sigma} = 0$, these categories are pairwise orthogonal. Moreover, since for all σ we have $e^{+}_{\sigma} = \sum_{\psi} e^{\psi}_{\sigma}$, we actually get a decomposition $\mathrm{Rep}^{0}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G) = \prod_{\psi} \mathrm{Rep}^{\psi}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. Correspondingly, we get a decomposition of ε_0 as a sum of pairwise orthogonal idempotents $\varepsilon_0 = \sum_{\psi} \varepsilon^{\psi}_0$ in $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$. Finally, identifying $\pi_0(\Psi_T)_p$ to the group of characters of $\pi_1(T)_{p-\mathrm{tors}}$, our constructions make it clear that the action of $\pi_0(\Psi_T)_p$ is given by $\psi \cdot \varepsilon^{\psi'}_0 = \varepsilon^{\psi\psi'}_0$. Since we already know that the action of $\pi_0(\Psi_T)_p$ is transitive on primitive idempotents, we conclude that each ε^{ψ}_0 has to be primitive, and that this action is simply transitive. □

3.6.3. *Remark.* Before turning to the dual side, we give an interpretation of the group $\Psi^{f}_{G}(\overline{\mathbb{Z}}[\frac{1}{p}])_p$ (through which the natural action of $\Psi_G(\overline{\mathbb{Z}}[\frac{1}{p}])$ on the set of idempotents in $\varepsilon_0\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G)$ factors) in terms of the group $\pi_0(\Psi_G)$ of connected components of Ψ_G. Indeed, more generally, for any diagonalizable group scheme $A = D(M)$ over $\overline{\mathbb{Z}}[\frac{1}{p}]$, we have an exact sequence $A^{\circ} \hookrightarrow A \twoheadrightarrow \pi_0(A)$ where $A^{\circ} = D(M/M_{p-\mathrm{tors}})$ is the "maximal" connected diagonalizable subgroup scheme of A and $\pi_0(A) = D(M_{p-\mathrm{tors}})$ is a (constant) finite étale diagonalizable group scheme. In the case $A = \Psi_G$, we thus see that $\pi_0(\Psi_G) = D(\pi_1(G)_{p-\mathrm{tors}})$ is the finite constant group scheme associated to the abstract group $\Psi^{f}_{G}(\overline{\mathbb{Z}}[\frac{1}{p}])_p = \mathrm{Hom}(\pi_1(G)_{p-\mathrm{tors}}, \overline{\mathbb{Z}}[\frac{1}{p}]^{\times})$, and we shall abuse a bit notation by writing

$$\pi_0(\Psi_G) = \Psi^{f}_{G}(\overline{\mathbb{Z}}[\tfrac{1}{p}])_p.$$

3.7. **The quasi-split case : dual group side.** We now explain how the description of primitive idempotents in the last theorem matches the parametrization of connected components of the space of tamely ramified Langlands parameters for G. We will use the definitions and notation from [DHKM20]. Let us denote by $\hat{\mathbf{G}}$ "the" dual group of $\mathbf{G}$, considered as a split reductive group scheme over $\overline{\mathbb{Z}}[\frac{1}{p}]$. Pick a pinning $\rho := (\hat{\mathbf{T}}, \hat{\mathbf{B}}, X = \sum_{\alpha\in\Delta^{\vee}} X_{\alpha})$ of $\hat{\mathbf{G}}$, whose underlying Borel pair is dual to a Borel pair $(\mathbf{T}, \mathbf{B})$ where $\mathbf{T}$ is as above (a maximally split maximal torus) and $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$ defined over F. Then the F-rational structure on $\mathbf{G}$ induces an action of W_F on the root datum of $\hat{\mathbf{G}}$, which induces in turn an action of W_F on $\hat{\mathbf{G}}$ preserving the pinning ρ.

3.7.1. *Remark.* Since the morphism $\mathbf{T}_{\mathrm{sc}} \longrightarrow \mathbf{T}$ is dual to the morphism $\hat{\mathbf{T}} \longrightarrow \hat{\mathbf{T}}_{\mathrm{ad}}$, we see that $\pi_1(\mathbf{G})$ is the group of characters of the center $Z(\hat{\mathbf{G}}) = \ker(\hat{\mathbf{T}} \longrightarrow \hat{\mathbf{T}}_{\mathrm{ad}})$. It follows in particular that

$$\Psi_G = (Z(\hat{\mathbf{G}})^{I_F})_{\mathsf{F}},$$

as group schemes over $\overline{\mathbb{Z}}[\frac{1}{p}]$. Then, using the last remark, we may slightly abusively identify

$$\Psi^{f}_{G}(\overline{\mathbb{Z}}[\tfrac{1}{p}]) = \pi_0((Z(\hat{\mathbf{G}})^{I_F})_{\mathsf{F}})$$

Let us now choose a topological generator σ of the tame inertia group I_F/P_F and denote by W^0_F the inverse image in W_F of the discrete subgroup of W_F/P_F generated by σ and Frobenius. According to [DHKM20, §1.2] , there is an affine scheme $\underline{Z}^1(W^0_F, \hat{\mathbf{G}})_{\mathrm{tame}}$ over $\overline{\mathbb{Z}}[\frac{1}{p}]$ that classifies 1-cocycles $W_F \longrightarrow \hat{\mathbf{G}}$ whose restriction to P_F is étale-locally conjugate to the trivial 1-cocycle $\phi = 1_{P_F} : P_F \longrightarrow \hat{\mathbf{G}}$. This affine scheme carries an action of $\hat{\mathbf{G}}$ over $\overline{\mathbb{Z}}[\frac{1}{p}]$ and factors as

$$\underline{Z}^1(W^0_F, \hat{\mathbf{G}})_{\mathrm{tame}} = \hat{\mathbf{G}} \times^{\hat{\mathbf{G}}^{P_F}} \underline{Z}^1(W^0_F, \hat{\mathbf{G}})_{1_{P_F}}$$

where $\underline{Z}^1(W_F^0, \hat{\mathbf{G}})_{1_{P_F}}$ is the closed subscheme of $\underline{Z}^1(W_F^0, \hat{\mathbf{G}})_{\mathrm{tame}}$ where the restriction of parameters to P_F is trivial, and where $\hat{\mathbf{G}}^{P_F}$ is the closed subgroup scheme of $\hat{\mathbf{G}}$ fixed by P_F. Note that, in the notation of [DHKM20], $\hat{\mathbf{G}}^{P_F}$ would be denoted $C_{\hat{\mathbf{G}}}(\phi)$ if $\phi = 1_{P_F}$. Since $\underline{Z}^1(W_F^0, \hat{\mathbf{G}})_{1_{P_F}} = \underline{Z}^1(W_F^0/P_F, \hat{\mathbf{G}}^{P_F})$, we get on quotient stacks

$$\underline{Z}^1(W_F^0, \hat{\mathbf{G}})_{\mathrm{tame}}/\hat{\mathbf{G}} = \underline{Z}^1(W_F^0/P_F, \hat{\mathbf{G}}^{P_F})/\hat{\mathbf{G}}^{P_F}. \tag{3.3}$$

We are interested in parametrizing the connected components of these stacks. According to Proposition A.13 and Theorem A.12 of [DHKM20], the $\overline{\mathbb{Z}}[\frac{1}{p}]$-group scheme $\hat{\mathbf{G}}^{P_F}$ has split reductive neutral component $\hat{\mathbf{G}}^{P_F,\circ}$ and finite constant $\pi_0(\hat{\mathbf{G}}^{P_F})$. We are going to prove that the fibers of the morphism

$$\underline{Z}^1(W_F^0/P_F, \hat{\mathbf{G}}^{P_F})/\hat{\mathbf{G}}^{P_F} \xrightarrow{\mu} H^1(W_F^0/P_F, \pi_0(\hat{\mathbf{G}}^{P_F})), \tag{3.4}$$

(whose target is a finite discrete scheme) are the connected components of its source. To this aim, observe that the diagonalizable group scheme $\underline{Z}^1(W_F^0/P_F, Z(\hat{\mathbf{G}})^{P_F})$ acts on the scheme $\underline{Z}^1(W_F^0/P_F, \hat{\mathbf{G}}^{P_F})$ by multiplication of cocycles, and this action is compatible with $\hat{\mathbf{G}}^{P_F}$-(twisted) conjugation on $\underline{Z}^1(W_F^0/P_F, \hat{\mathbf{G}}^{P_F})$. Furthermore, the map μ is equivariant if we let $\underline{Z}^1(W_F^0/P_F, Z(\hat{\mathbf{G}})^{P_F})$ act on $H^1(W_F^0/P_F, \pi_0(\hat{\mathbf{G}}^{P_F}))$ through $H^1(W_F^0/P_F, \pi_0(Z(\hat{\mathbf{G}})^{P_F}))$.

3.7.2. **Lemma.** *With the foregoing notation :*

(1) *The natural map $\pi_0(\hat{\mathbf{T}}^{P_F}) \longrightarrow \pi_0(\hat{\mathbf{G}}^{P_F})$ is a bijection. In particular, $\pi_0(\hat{\mathbf{G}}^{P_F})$ is an abelian p-group.*

(2) *The natural map*
$$\pi_0(Z(\hat{\mathbf{G}})^{I_F})_{\mathsf{F}} = H^1(\langle \mathsf{F} \rangle, \pi_0(Z(\hat{\mathbf{G}})^{I_F})) \longrightarrow H^1(W_F^0/P_F, \pi_0(Z(\hat{\mathbf{G}})^{P_F}))$$
is an isomorphism.

(3) *Similarly, we have an isomorphism $\pi_0(\hat{\mathbf{T}}^{I_F})_{\mathsf{F}} \xrightarrow{\sim} H^1(W_F^0/P_F, \pi_0(\hat{\mathbf{T}}^{P_F}))$.*

(4) *The natural map $H^1(W_F^0/P_F, \pi_0(Z(\hat{\mathbf{G}})^{P_F})) \longrightarrow H^1(W_F^0/P_F, \pi_0(\hat{\mathbf{T}}^{P_F}))$ is surjective.*

Proof. (1) This is Proposition 4.1 d) of [Hai15].

(2) Recall first that $\pi_0(Z(\hat{\mathbf{G}})^{P_F})$ is a finite abelian p-group, and the action of I_F on it is through a cyclic p'-group. It follows that $H^1(I_F^0/P_F, \pi_0(Z(\hat{\mathbf{G}})^{P_F})) = \{1\}$ and therefore the map $H^1(\langle \mathsf{F} \rangle, \pi_0(Z(\hat{\mathbf{G}})^{P_F})^{I_F}) \longrightarrow H^1(W_F^0/P_F, \pi_0(Z(\hat{\mathbf{G}})^{P_F}))$ is an isomorphism. So it remains to see that $\pi_0(Z(\hat{\mathbf{G}})^{P_F})^{I_F} = \pi_0(Z(\hat{\mathbf{G}})^{I_F})$, which follows from the fact that $(X^*(Z(\hat{\mathbf{G}}))_{P_F,p-\mathrm{tors}})_{I_F} = X^*(Z(\hat{\mathbf{G}}))_{I_F,p-\mathrm{tors}}$ since, as above, the action of I_F on $X^*(Z(\hat{\mathbf{G}}))_{P_F}$ is through a cyclic p'-group.

(3) Apply (2) to $\hat{\mathbf{T}}$ instead of $\hat{\mathbf{G}}$.

(4) By (2) and (3), it suffices to prove surjectivity of $\pi_0(Z(\hat{\mathbf{G}})^{I_F}) \longrightarrow \pi_0(\hat{\mathbf{T}}^{I_F})$, i.e. injectivity of $X^*(\hat{\mathbf{T}})_{I_F,p-\mathrm{tors}} \longrightarrow X^*(Z(\hat{\mathbf{G}}))_{I_F,p-\mathrm{tors}}$. For this we start from the exact sequence

$$X^*(\hat{\mathbf{T}}_{\mathrm{ad}})_{I_F} \longrightarrow X^*(\hat{\mathbf{T}})_{I_F} \longrightarrow X^*(Z(\hat{\mathbf{G}}))_{I_F} \longrightarrow 0$$

and we observe that, since $X^*(\hat{\mathbf{T}}_{\mathrm{ad}})$ has a basis permuted by I_F (given by simple roots), its co-invariants $X^*(\hat{\mathbf{T}}_{\mathrm{ad}})_{I_F}$ are a free abelian group. Moreover, since the above sequence is exact on the left once we tensor it by $\mathbb{Q}$, it follows that $X^*(\hat{\mathbf{T}}_{\mathrm{ad}})_{I_F} \longrightarrow X^*(\hat{\mathbf{T}})_{I_F}$ is injective, hence $X^*(\hat{\mathbf{T}})_{I_F,\mathrm{tors}} \longrightarrow X^*(Z(\hat{\mathbf{G}}))_{I_F,\mathrm{tors}}$ is injective too. $\square$

The following theorem, together with the identification $\Psi_T^f(\overline{\mathbb{Z}}[\frac{1}{p}]) = \pi_0((\hat{\mathbf{T}}^{I_F})_{\mathsf{F}}) = \pi_0(\hat{\mathbf{T}}^{I_F})_{\mathsf{F}}$ of Remark 3.7.1 is the dual companion of Theorem 3.6.2.

3.7.3. **Theorem.** *The connected components of $Z^1(W_F^0,\hat{\mathbf{G}})_{\rm tame}/\hat{\mathbf{G}}$ are the fibers of the map μ of (3.4) through the identification (3.3) :*

$$\underline{Z}^1(W_F^0/P_F,\hat{\mathbf{G}}^{P_F})/\hat{\mathbf{G}}^{P_F}=\underline{Z}^1(W_F^0,\hat{\mathbf{G}})_{\rm tame}/\hat{\mathbf{G}}.$$

Moreover, the action of $\underline{Z}^1(W_F^0/P_F,Z(\hat{\mathbf{G}})^{P_F})$ on $\underline{Z}^1(W_F^0/P_F,\hat{\mathbf{G}}^{P_F})$ induces a simply transitive action of $\pi_0(\hat{\mathbf{T}}^{I_F})_{\mathsf{F}}$ on connected components of $\underline{Z}^1(W_F^0,\hat{\mathbf{G}})_{\rm tame}/\hat{\mathbf{G}}$.

Proof. By construction, the action of $Z^1(W_F^0/P_F,\pi_0(Z(\hat{\mathbf{G}})^{P_F}))$ on $Z^1(W_F^0/P_F,\pi_0(\hat{\mathbf{G}}^{P_F}))$ induces an action of $H^1(W_F^0/P_F,\pi_0(Z(\hat{\mathbf{G}})^{P_F}))$ on the set of fibers of the map μ. By (1) and (4) of the last lemma, the latter action is transitive, and actually factors over a simply transitive action of $H^1(W_F^0/P_F,\pi_0(\hat{\mathbf{T}}^{P_F}))=\pi_0(\hat{\mathbf{T}}^{I_F})_{\mathsf{F}}$. So, to prove the theorem, it suffices to prove that one fiber of μ is connected. The fiber $\mu^{-1}(1)$ of the trivial cohomology class is $Z^1(W_F^0/P_F,\hat{\mathbf{G}}^{P_F,\circ})/N\hat{\mathbf{G}}^{P_F,\circ}$, where $N\hat{\mathbf{G}}^{P_F,\circ}=\{g\in\hat{\mathbf{G}}^{P_F},g^{-1}\sigma(g)\in\hat{\mathbf{G}}^{P_F,\circ},g^{-1}\mathsf{F}(g)\in\hat{\mathbf{G}}^{P_F,\circ}\}$. So we are left with proving that $Z^1(W_F^0/P_F,\hat{\mathbf{G}}^{P_F,\circ})$ is connected. In order to apply [DHKM20, Thm. 4.29], we need to show that the action of W_F/I_F on $\hat{\mathbf{G}}^{P_F,\circ}$ fixes a pinning. Thanks to [Hai15, Prop 4.1.(a)], we know that $(\hat{\mathbf{B}}^{P_F,\circ},\hat{\mathbf{T}}^{P_F,\circ})$ is a Borel pair of $\hat{\mathbf{G}}^{P_F,\circ}$ (over $\overline{\mathbb{Z}}[\frac{1}{p}]$) and even that $(\hat{\mathbf{B}}^{P_F,\circ},\hat{\mathbf{T}}^{P_F,\circ},X)$ is a pinning of $\hat{\mathbf{G}}^{P_F,\circ}$, *at least over* $\overline{\mathbb{Z}}[\frac{1}{2p}]$. Note that this Borel pair and this pinning are clearly stable under W_F/I_F. So, when $p=2$, we are done. On the other hand, as explained in the proof of [Hai15, Prop 4.1], the failure for X to provide a pinning of $\hat{\mathbf{G}}^{P_F,\circ}$ in characteristic 2 only happens when an orbit of simple roots under P_F contains two roots that add up to a root. In this case, the orbit must have even order, and this can't happen if p is odd. So, in all cases, $(\hat{\mathbf{B}}^{P_F,\circ},\hat{\mathbf{T}}^{P_F,\circ},X)$ is a pinning of $\hat{\mathbf{G}}^{P_F,\circ}$ and we get connectedness of $Z^1(W_F^0/P_F,\hat{\mathbf{G}}^{P_F,\circ})$ from [DHKM20, Thm. 4.29]. □

Finally, Theorem 1.0.4 follows from Theorems 3.7.3 and 3.6.2 through the identification of Remark 3.7.1, once we choose base points in the respective sets of connected components. Any "natural" such choice should be compatible with parabolic induction from the minimal Levi subgroup $\mathbf{T}$ of $\mathbf{G}$. But the Langlands correspondence for tori tells us that the principal block of $\mathrm{Rep}^0_{\overline{\mathbb{Z}}[\frac{1}{p}]}(T)$ (i.e. the one that contains the trivial representation) should match the principal component of $\underline{Z}^1(W_F^0,\hat{\mathbf{T}})_{\rm tame}$ (i.e. the one that contains the trivial parameter).

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